Name of college - S. S. college, J-Bad

Deptt - Mathematics

Topic - Problem based on absolutely convergent and conditionally convergent series

Class - B.Sc II

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Study Material :->

Conditionally Convergent Series :->

If the series Σu_n is convergent but the series $\Sigma|u_n|$ is divergent, then the series Σu_n is said to be conditionally convergent or semi convergent series.

In other words.

If an infinite series Σu_n is convergent but is not absolutely convergent, then this Σu_n is said to be conditionally convergent series

Remarks 1 :-> The series formed by taking positive terms of a conditionally convergent series is a divergent series.

Remarks 2: → The Problem of Testing whether or not a series Σu_n is absolutely convergent reduces to Testing for convergence the Series of +ve Terms $\Sigma |u_n|$

Problem :→ Show that the series

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$ Converges

absolutely for all values of x.

Solution :→ Let $\Sigma u_n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$ ①

Then $u_n = \frac{x^n}{\lfloor n}$ (On Leaving First Term.

and $u_{n+1} = \frac{x^{n+1}}{\lfloor n+1}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{x^n}{\lfloor n} \cdot \frac{\lfloor n+1}{x^{n+1}} = \frac{n+1}{x}$

$\therefore \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{|x|} = \infty$ for all non-zero values of x

$\therefore$ By ratio Test

Σu_n is convergent for all non-zero values of x.

If however $x = 0$

the series Σu_n from ① is clearly convergent

Hence the Given Exponential Series Σu_n is absolutely convergent for all values of x.

Problem 2. Test the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$$

for absolute Convergence.

Solⁿ :→

Let $\sum u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ to ∞

then $u_n = \frac{(-1)^{n-1}}{n} x^n$

and $u_{n+1} = \frac{(-1)^n x^{n+1}}{n+1}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{(-1)^{n-1} x^n}{n} \times \frac{(n+1)}{(-1)^n x^{n+1}}$

$= -\left(\frac{n+1}{nx}\right) = -\frac{n+1}{n} \cdot \frac{1}{x}$

$= -\left(1 + \frac{1}{n}\right) \cdot \frac{1}{x}$

$\therefore \left|\frac{u_n}{u_{n+1}}\right| = \left|\left(1 + \frac{1}{n}\right)\right| \cdot \left|\frac{1}{x}\right|$

$= \left|1 + \frac{1}{n}\right| \cdot \frac{1}{|x|}$

$\therefore \lim_{n \to \infty} \left|\frac{u_n}{u_{n+1}}\right| = \frac{1}{|x|}$

By ratio Test, we find that the series $\sum |u_n|$ is convergent or divergent according as

$\frac{1}{|x|} > 1$ or

$\frac{1}{|x|} < 1$

i.e $|x| < 1$ convergent

or

$|x| > 1$ divergent

If $x < 0$ then $|u_n| = -u_n$
and therefore Σu_n is convergent or divergent and if and only if $\Sigma|u_n|$ is convergent or divergent.

Since $\Sigma|u_n|$ is divergent if $x < -1$

So Σu_n is divergent if $x < -1$

If $x = 1$ we have

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots = \log 2$$

and is convergent series

If $x = -1$ we have $\Sigma u_n = -(1 + 1/2 + 1/3 + 1/4 \cdots)$

$= -\Sigma \frac{1}{n}$ which is divergent series

As $\Sigma \frac{1}{n^p}$ is divergent if $p \leq 1$

If $x > 1$ $\lim_{n \to \infty} u_n \neq 0$

So Σu_n is not convergent series if $x > 1$

Hence Σu_n converges if and only if

$-1 < x \leq 1$